Remark: The dyadic square function is often expressed as $S_{\mathcal{D}}^2\varphi = \sum_{I\in\mathcal{D}} (\Delta_I\varphi)^2 \mathbb{1}_I$ where Δ_I is the martingale difference operator:

$$\Delta_I\varphi := \sum_{K\in ch(I)} \mathbb{1}_K(\langle\varphi\rangle_K - \langle\varphi\rangle_I)$$

$$= \mathbb{1}_{I_+}(\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_I) + \mathbb{1}_{I_-}(\langle\varphi\rangle_{I_-} - \langle\varphi\rangle_I)$$

$$= \mathbb{1}_{I_+}\left(\frac{\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_{I_-}}{2}\right) + \mathbb{1}_{I_-}\left(\frac{\langle\varphi\rangle_{I_-} - \langle\varphi\rangle_{I_+}}{2}\right)$$

$$\langle\varphi\rangle_I = \frac{\langle\varphi\rangle_{I_-} + \langle\varphi\rangle_{I_+}}{2}$$

$$= \mathbb{1}_{I_+}(\varphi, h_I)\frac{1}{\sqrt{|I|}} - \mathbb{1}_{I_-}(\varphi, h_I)\frac{1}{\sqrt{|I|}}$$

$$(\varphi, h_I) = \sqrt{|I|}\,\frac{\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_{I_-}}{2}$$

$$= (\varphi, h_I)h_I$$

$$\Rightarrow \sum_{I\in\mathcal{D}} (\Delta_I\varphi)^2\mathbb{1}_I = \sum_{I\in\mathcal{D}} (\varphi, h_I)^2\frac{\mathbb{1}_I}{|I|}.$$

$$\boxed{\Delta_I\varphi = (\varphi, h_I)h_I}$$

A simple but useful inequality:

$$\boxed{\langle\varphi\rangle_J^2 + \Delta_J^2\varphi \leq \langle\varphi^2\rangle_J} \quad (*)$$

$$\Delta_J^2\varphi = \left(\frac{\langle\varphi\rangle_{J_+} - \langle\varphi\rangle_{J_-}}{2}\right)^2\mathbb{1}_J$$

$$\Rightarrow \langle\varphi\rangle_J^2 + \Delta_J^2\varphi = \left(\frac{\langle\varphi\rangle_{J_+} + \langle\varphi\rangle_{J_-}}{2}\right)^2 + \left(\frac{\langle\varphi\rangle_{J_+} - \langle\varphi\rangle_{J_-}}{2}\right)^2$$

$$= \frac{1}{2}\left(\langle\varphi\rangle_{J_-}^2 + \langle\varphi\rangle_{J_+}^2\right) \quad \leftarrow \text{Hölder: } \langle\varphi\rangle_{J_-}^2 \leq \langle\varphi^2\rangle_{J_-}$$

$$\leq \frac{1}{2}\left(\langle\varphi^2\rangle_{J_-} + \langle\varphi^2\rangle_{J_+}\right)$$

$$= \langle\varphi^2\rangle_J.$$

B is the least supersolution: $\langle\varphi\rangle_J^2 + \Delta_J^2\varphi \leq \langle\varphi^2\rangle_J \quad (*)$

Def: A supersolution for this problem is any continuous $u:\Omega \to [0,1]$ that satisfies:

① **Main Inequality:** $u\left(f, F, \lambda + \left(\frac{f_+ - f_-}{2}\right)^2\right) \geq \frac{1}{2}\left(u(f_-, F_-, \lambda) + u(f_+, F_+, \lambda)\right)$

② **Obstacle Condition:** $u(f, F, \lambda) = 1, \; \forall\, \lambda \leq F - f^2$

THM: $\boxed{B \leq u}$ for any supersolution u.

Proof: We need to show that, if u is any supersolution and φ is any function on J with $\langle\varphi\rangle_J = f$, $\langle\varphi^2\rangle_J = F$, we have

$$u(f,F,\lambda) \geq \frac{1}{|J|}|E_\varphi| \quad (1) \quad \text{where} \quad \boxed{E_\varphi := \{x \in J : S_J^2\varphi(x) > \lambda\}}$$

It suffices to prove (1) for functions φ with finite Haar expansion.

So assume that $\boxed{\varphi = f\mathbb{1}_J + \sum_{\substack{I \subset J \\ |I| \geq |J|2^{-N}}} (\varphi, h_I)h_I}$ for some dyadic level $N \geq 0$.

For every $I \subset J$, let $f_I := \langle\varphi\rangle_I$; $F_I := \langle\varphi^2\rangle_I$; $\lambda_I := \lambda - \sum_{K: I \subsetneq K \subset J} \Delta_K^2\varphi$.

Then note that: $f = f_J = \frac{1}{2}(f_{J_-} + f_{J_+})$; $F = F_J = \frac{1}{2}(F_{J_+} + F_{J_-})$; $\lambda_J = \lambda$.

➡ If $\boxed{\lambda < \Delta_J^2\varphi}$ then by $(*)$: $\lambda < \Delta_J^2\varphi \leq F - f^2 \underset{OC}{\Rightarrow} u(f,F,\lambda) = 1 \geq \frac{|E\varphi|}{|J|}$ done.

➡ Otherwise: Suppose $\boxed{\lambda_{J_+} = \lambda_{J_-} = \lambda - \Delta_J^2\varphi > 0}$ => Apply MI to obtain:

$$|J|\,u(f,F,\lambda) \geq |J_-|\,u(f_{J_-}, F_{J_-}, \lambda_{J_-}) + |J_+|\,u(f_{J_+}, F_{J_+}, \lambda_{J_+}) \qquad (2)$$

➡ If $\boxed{\lambda < \Delta_J^2\varphi + \Delta_{J_+}^2\varphi}$ then $\lambda_{J_+} = \lambda - \Delta_J^2\varphi \leq \Delta_{J_+}^2\varphi \leq F_{J_+} - f_{J_+}^2$ by $(*)$ and so by the OC: $u(f_{J_+}, F_{J_+}, \lambda_{J_+}) = 1$. Then (2) becomes:

$$|J|\,u(f,F,\lambda) \geq |J_-|\,u(f_{J_-}, F_{J_-}, \lambda_{J_-}) + |J_+|$$

and if we iterate further, we do so only on J_-. Remark that:

$$\forall\, I \in J_{(N)},\, I \subset J_+ : \lambda \leq \Delta_J^2\varphi + \Delta_{J_+}^2\varphi \leq \Delta_J^2\varphi + \Delta_{J_+}^2\varphi + \ldots + \Delta_{I^{(1)}}^2\varphi \Rightarrow$$

$$\Rightarrow \boxed{\lambda_I \leq 0} \; \forall\, I \in J_{(N)},\, I \subset J_+$$

➡ Otherwise, iterate the J_+ term further, with $\lambda_{J_{+\pm}} = \lambda_{J_{+\mp}} = \lambda - \Delta_J^2\varphi - \Delta_{J_+}^2\varphi > 0$

(Same analysis for J_-)

Continue down to level N:

$$|J|\,u(f,F,\lambda) \geq \sum_{\substack{I\in J_{(N)}\\ \lambda_I>0}} |I|\,u(f_I,F_I,\lambda_I) + \sum_{\substack{I\in J_{(N)}\\ \lambda_I\leq 0}} |I| \qquad (3)$$

Let $I\in J_{(N)}$. Then $S_J^2\varphi(x) = \Delta_I^2\varphi + \Delta_{I^{(1)}}^2\varphi + \ldots + \Delta_J^2\varphi$, $\forall x\in I$, so

$$I\subset E_\varphi \iff \lambda \leq \Delta_I^2\varphi + \ldots + \Delta_J^2\varphi \iff \lambda_I \leq \Delta_I^2\varphi$$

$$\boxed{I\in J_{(N)}:\ I\subset E_\varphi \iff \lambda_I \leq \Delta_I^2\varphi}$$

So if $I\subset E_\varphi$ and $\lambda_I>0$, then $\lambda_I < \Delta_I^2\varphi \leq F_I - f_I^2$ by $(*)$, so by $\underline{OC}$: $|I|\,u(f_I,F_I,\lambda_I) = |I|$

$\Rightarrow$ (3) gives that:

$$|J|\,u(f,F,\lambda) \geq \sum_{\substack{I\in J_{(N)}\\ \lambda_I>0\\ I\subset E_\varphi}} |I|\,u(f_I,F_I,\lambda_I) + \underbrace{\sum_{\substack{I\in J_{(N)}\\ \lambda_I\leq 0}} |I|}_{\substack{I\subset E_\varphi \\ \text{automatically} \\ \text{b/c } \lambda_I\leq 0\leq \Delta_I^2\varphi}}$$

$$= \sum_{\substack{I\in J_{(N)}\\ I\subset E_\varphi}} |I| = |E_\varphi| \quad /\!\!/$$

$\blacksquare$

Finding $B(f, F, \lambda)$:

Same as in the $L^1 \to L^{1,\infty}$ case

→ The main inequality in differential form: $\begin{pmatrix} B_{ff} - 2B_\lambda & B_{fF} \\ B_{fF} & B_{FF} \end{pmatrix} \leq 0$

→ Using homogeneities:

$$\boxed{B(f, F, \lambda) = \theta(\tau)}, \text{ where } \boxed{\tau = \frac{F - f^2}{\lambda}}$$

→ The obstacle condition gives: $B(f,F,\lambda) = 1$, $\forall \frac{F-f^2}{\lambda} \geq 1 \Rightarrow \boxed{\theta(\tau) = 1, \forall \tau \geq 1}$.

→ The boundary condition $B(f, f^2, \lambda) = 0$ gives that $\boxed{\theta(0) = 0}$.

→ The derivatives:

$$B_f = \theta'(\tau)\frac{-2f}{\lambda} \Rightarrow B_{ff} = \theta''(\tau)\frac{4f^2}{\lambda^2} - \theta'(\tau)\frac{2}{\lambda}$$

$$\Rightarrow B_{fF} = \theta''(\tau)\frac{-2f}{\lambda^2}$$

$$B_\lambda = \theta'(\tau)\frac{-(F-f^2)}{\lambda^2}$$

$$B_F = \theta'(\tau)\frac{1}{\lambda} \Rightarrow B_{FF} = \theta''(\tau)\frac{1}{\lambda^2}$$

⇒ The matrix inequality becomes

$$\begin{pmatrix} \frac{4f^2}{\lambda^2}\theta''(\tau) - \frac{2}{\lambda}\theta'(\tau) + \frac{2}{\lambda}\underbrace{\frac{F-f^2}{\lambda}}_{\tau}\theta'(\tau) & -\frac{2f}{\lambda^2}\theta''(\tau) \\ -\frac{2f}{\lambda^2}\theta''(\tau) & \frac{1}{\lambda^2}\theta''(\tau) \end{pmatrix} \leq 0$$

$$\boxed{R_\theta = \begin{pmatrix} 4f^2\theta''(\tau) + 2\lambda(\tau - 1)\theta'(\tau) & 2f\theta''(\tau) \\ 2f\theta''(\tau) & \theta''(\tau) \end{pmatrix} \leq 0 \qquad (*)}$$

Remarks:

→ We only need to consider $(*)$ for $0 \leq \tau \leq 1$, since $\theta(\tau) = 1, \forall \tau \geq 1$.

→ $\boxed{\theta'(\tau) \geq 0}$: For $f > 0$, we know B is decreasing in f, i.e.

$f > 0 \Rightarrow B_f \leq 0 \Rightarrow \theta'(\tau) = -\frac{\lambda}{2f} B_f \geq 0$.

→ $\boxed{\theta''(\tau) \leq 0}$: Recall that B is concave in F, so $\theta''(\tau) = \lambda^2 B_{FF} \leq 0$.

⇒ $(*)$ now only requires that $\boxed{\det(R_\theta) \geq 0}$

→ $\det(R_\theta) = \cancel{4f^2(\theta''(\tau))^2} + 2\lambda(\tau-1)\,\theta'(\tau)\,\theta''(\tau) - \cancel{4f^2(\theta''(\tau))^2}$

$$\det(R_\theta) \geq 0 \iff \boxed{(\tau-1)\,\theta'(\tau)\,\theta''(\tau) \geq 0}$$

→ Remark that this <u>always holds</u>: for $0 \leq \tau \leq 1$,

$$\left.\begin{matrix} (\tau-1) \leq 0 \\ \theta'(\tau) \geq 0 \\ \theta''(\tau) \leq 0 \end{matrix}\right\} \Rightarrow \det(R_\theta) \geq 0.$$

→ Setting $\det(R_\theta) = 0$ requires that $\boxed{\theta''(\tau) = 0}$ (we cannot have $\theta'(\tau) = 0$, for this would make $\theta \equiv C$).

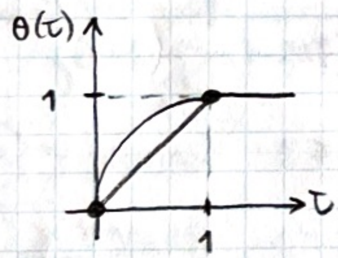
$\Rightarrow$ $\theta(\tau) = c\tau + d$ for $\tau \in [0,1]$.

$\theta(0) = 0 \Rightarrow d = 0$; $\quad \theta(1) = 1 \Rightarrow c = 1$

⇒ <u>Propose the function</u>:

$$\boxed{\tilde{\theta}(\tau) = \begin{cases} \tau, & \text{for } \tau \in [0,1] \\ 1, & \text{for } \tau \geq 1 \end{cases}} \Rightarrow \boxed{\tilde{B}(f,F,\lambda) = \begin{cases} \dfrac{F-f^2}{\lambda}, & \text{for } \lambda \geq F-f^2 \\ 1, & \text{for } \lambda \leq F-f^2 \end{cases}}$$

→ Since $\theta''(\tau) \leq 0$, so θ is <u>concave</u>, it lies above the secant line on $\tau \in [0,1]$



Since $\theta(0) = 0$ and $\theta(1) = 1$, this secant line is just $y = \tau$.

$\Rightarrow \theta(\tau) \geq \tau$ on $\tau \in [0,1]$

$\Rightarrow \theta(\tau) \geq \tilde{\theta}(\tau)$, or $\boxed{B \geq \tilde{B}}$

→ Conversely, we show that <u>$\tilde{B}$ is a supersolution</u> (check Mi), and so $\boxed{B \leq \tilde{B}}$.

(See next for Mi check).

$\Rightarrow \boxed{\boxed{B = \tilde{B}}}$

Check Mi: $\widetilde{\mathbb{B}}(\mathfrak{f}, F, \lambda+a^2) \geq \frac{1}{2}\left(\widetilde{\mathbb{B}}(\mathfrak{f}-a, F-b, \lambda) + \widetilde{\mathbb{B}}(\mathfrak{f}+a, F+b, \lambda)\right)$

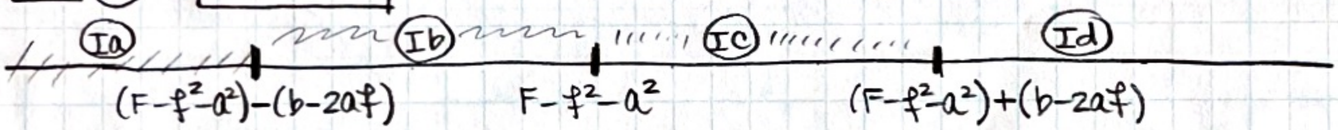
$$\widetilde{\mathbb{B}}(\mathfrak{f}, F, \lambda+a^2) = \begin{cases} \dfrac{F-\mathfrak{f}^2}{\lambda+a^2}, & \lambda+a^2 \geq F-\mathfrak{f}^2, \text{ or } \lambda \geq F-\mathfrak{f}^2-a^2 \\ 1, & \lambda+a^2 \leq F-\mathfrak{f}^2, \text{ or } \lambda \leq F-\mathfrak{f}^2-a^2 \end{cases}$$

$$\widetilde{\mathbb{B}}(\mathfrak{f}-a, F-b, \lambda) = \begin{cases} \dfrac{F-b-(\mathfrak{f}-a)^2}{\lambda}, & \lambda \geq F-b-(\mathfrak{f}-a)^2 = F-\mathfrak{f}^2-a^2-b+2a\mathfrak{f} \\ 1, & \lambda \leq F-b-(\mathfrak{f}-a)^2 = (F-\mathfrak{f}^2-a^2)-(b-2a\mathfrak{f}) \end{cases}$$

$$\widetilde{\mathbb{B}}(\mathfrak{f}+a, F+b, \lambda) = \begin{cases} \dfrac{F+b-(\mathfrak{f}+a)^2}{\lambda}, & \lambda \geq F+b-(\mathfrak{f}+a)^2 = F-\mathfrak{f}^2-a^2+b-2a\mathfrak{f} \\ 1, & \lambda \leq F-\mathfrak{f}^2-a^2+b-2a\mathfrak{f} \end{cases}$$

Must arrange on axis: $(F-\mathfrak{f}^2-a^2)$, $(F-\mathfrak{f}^2-a^2)-(b-2a\mathfrak{f})$, $(F-\mathfrak{f}^2-a^2)+(b-2a\mathfrak{f})$.

Case (I): $\boxed{b-2a\mathfrak{f} \geq 0}$ (The other case should be symmetric).



(Ia) Mi becomes: $1 \geq \frac{1}{2}(1+1)$ True.

(Ib) Mi becomes: $1 \geq \frac{1}{2}\left(\frac{F-b-(\mathfrak{f}-a)^2}{\lambda}+1\right)$; $1 \geq \frac{F-b-(\mathfrak{f}-a)^2}{\lambda}$; $\lambda \geq (F-\mathfrak{f}^2-a^2)-(b-2a\mathfrak{f})$ yes

(Ic) Mi becomes: $\frac{F-\mathfrak{f}^2}{\lambda+a^2} \geq \frac{1}{2}\left(\frac{F-b-(\mathfrak{f}-a)^2}{\lambda}+1\right)$

$$\frac{2(F-\mathfrak{f}^2)}{\lambda+a^2} \geq \frac{F-b-(\mathfrak{f}-a)^2+\lambda}{\lambda} = \frac{F-\mathfrak{f}^2-a^2-(b-2a\mathfrak{f})+\lambda}{\lambda}$$

$$2\lambda F - 2\lambda\mathfrak{f}^2 \geq (\lambda+a^2)\left(F-\mathfrak{f}^2-a^2-(b-2a\mathfrak{f})+\lambda\right)$$

$F-\mathfrak{f}^2-a^2-(b-2a\mathfrak{f})+\lambda \leq 2(F-\mathfrak{f}^2-a^2)$ Yes!

$(\lambda+a^2)(F-\mathfrak{f}^2-a^2-(b-2a\mathfrak{f})+\lambda) \leq 2(\lambda+a^2)(F-\mathfrak{f}^2-a^2) \overset{?}{\leq} 2\lambda(F-\mathfrak{f}^2)$

$$\cancel{\lambda F} - \cancel{\lambda\mathfrak{f}^2} - \lambda a^2 + a^2F - a^2\mathfrak{f}^2 - a^4 \leq \cancel{\lambda F} - \cancel{\lambda\mathfrak{f}^2}$$

$$-\lambda + F - \mathfrak{f}^2 - a^2 \leq 0$$

$$F-\mathfrak{f}^2-a^2 \leq \lambda \quad \text{Yes}$$

Id) Mi becomes:

$$\frac{F-f^2}{\lambda+a^2} \geq \frac{1}{2}\left(\frac{F-b-(f-a)^2}{\lambda} + \frac{F+b-(f+a)^2}{\lambda}\right)$$

$$= \frac{F\cancel{-b}-f^2\cancel{+2af}-a^2+F\cancel{+b}-f^2-a^2\cancel{-2af}}{2\lambda}$$

$$= \frac{F-f^2-a^2}{\lambda}$$

$$\cancel{\lambda F}-\cancel{\lambda f^2} \geq \cancel{\lambda F}-\cancel{\lambda f^2}-\lambda a^2+a^2F-a^2f^2-a^4$$

$$0 \geq -\lambda+F-f^2-a^2$$

$$\lambda \geq F-f^2-a^2$$ Yes.